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Advanced Linear Algebra (MA 409)
Problem Sheet - 26

Unitary and Orthogonal Operators and Their Matrices

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1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

   (a) Every unitary operator is normal.

   (b) Every orthogonal operator is diagonalizable.

   (c) A matrix is unitary if and only if it is invertible.

   (d) If two matrices are unitarily equivalent, then they are also similar.

   (e) The sum of unitary matrices is unitary.

   (f) The adjoint of a unitary operator is unitary.

   (g) If T is an orthogonal operator on V, then $[T]_\beta$ is an orthogonal matrix for any ordered basis β for V.

   (h) If all the eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

   (i) A linear operator may preserve the norm, but not the inner product.

2. For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

   a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$

   d) $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ e) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

3. Prove that the composite of unitary [orthogonal] operators is unitary [orthogonal].

4. For $z \in \mathbb{C}$, define $T_z : \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Characterize those z for which T_z is normal, self-adjoint, or unitary.

5. Which of the following pairs of matrices are unitarily equivalent?

   a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

   c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$

e) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

6. Let V be the inner product space of complex-valued continuous functions on [0,1] with the inner product
$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\, dt.$$
Let $h \in V$, and define $T : V \to V$ by $T(f) = hf$. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for $0 \le t \le 1$.

7. Prove that if T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary *square root:* that is, there exists a unitary operator U such that $T = U^2$.

8. Let T be a self-adjoint linear operator on a finite-dimensional inner product space. Prove that $(T + iI)(T - iI)^{-1}$ is unitary.

9. Let U be a linear operator on a finite-dimensional inner product space V. If $\|U(x)\| = \|x\|$ for all x in some orthonormal basis for V, must U be unitary? Justify your answer with a proof or a counterexample.

10. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that
$$tr(A) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad tr(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2.$$
where the λ_i's are the (not necessarily distinct) eigenvalues of A.

11. Find an orthogonal matrix whose first row is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

12. Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that
$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$
where the λ_i's are the (not necessarily distinct) eigenvalues of A.

13. Suppose that A and B are diagonalizable matrices. Prove or disprove that A is similar to B if and only if A and B are unitarily equivalent.

14. Prove that if A and B are unitarily equivalent matrices, then A is positive definite [semidefinite] if and only if B is positive definite [semidefinite].

15. Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that

    (a) $U(W) = W$;

    (b) $W^\perp$ is U-invariant.

    Contrast (b) with Exercise 16.

16. Find an example of a unitary operator U on an inner product space and a U-invariant subspace W such that $W^{\perp}$ is not U-invariant.

17. Prove that a matrix that is both unitary and upper triangular must be a diagonal matrix.

18. Show that "is unitarily equivalent to"is an equivalence relation on $M_{n\times n}(\mathbb{C})$.

19. Let W be a finite-dimensional subspace of an inner product space V and $V = W \oplus W^{\perp}$. Define $U : V \to V$ by $U(v_1 + v_2) = v_1 - v_2$, where $v_1 \in W$ and $v_2 \in W^{\perp}$. Prove that U is a self-adjoint unitary operator.

20. Let V be a finite-dimensional inner product space. A linear operator U on V is called a **partial isometry** if there exists a subspace W of V such that $\|U(x)\| = \|x\|$ for all $x \in W$ and $U(x) = 0$ for all $x \in W^{\perp}$. Observe that W need not be U-invariant. Suppose that U is such an operator and $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for W. Prove the following results.

    (a) $\langle U(x), U(y) \rangle = \langle x, y \rangle$ for all $x, y \in W$.

    (b) $\{U(v_1), U(v_2), \ldots, U(v_k)\}$ is an orthonormal basis for $R(U)$.

    (c) There exists an orthonormal basis γ for V such that the first k columns of $[U]_{\gamma}$ form an orthonormal set and the remaining columns are zero.

    (d) Let $\{w_1, w_2, \ldots, w_j\}$ be an orthonormal basis for $R(U)^{\perp}$ and

    $$\beta = \left\{ U(v_1), U(v_2), \ldots, U(v_k), w_1, \ldots, w_j \right\}.$$

    Then β is an orthonormal basis for V.

    (e) Let T be the linear operator on V that satisfies $T(U(v_i)) = v_i$ $(1 \le i \le k)$ and $T(w_i) = 0$ $(1 \le i \le j)$. Then T is well defined, and $T = U^*$.
    Hint: Show that $\langle U(x), y \rangle = \langle x, T(y) \rangle$ for all $x, y \in \beta$. There are four cases.

    (f) U^* is a partial isometry.

21. Let A and B be $n \times n$ matrices that are unitarily equivalent.

    (a) Prove that $tr(A^*A) = tr(B^*B)$.

    (b) Use (a) to prove that

    $$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2.$$

    (c) Use (b) to show that the matrices

    $$\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$$

    are *not* unitarily equivalent.

22. Let V be a real inner product space.

    (a) Prove that any translation on V is a rigid motion.

    (b) Prove that the composite of any two rigid motions on V is a rigid motion on V.

23. **Theorem 6.22 :** Let $f : V \to V$ be a rigid motion on a finite-dimensional ral inner product space V. Then there exists a unique orthogonal operator T on V and a unique translation g on V such that $f = g \circ T$.

    Prove the following variation of Theorem 6.22: If $f : V \to V$ is a rigid motion on a finite-dimensional real inner product space V, then there exists a unique orthogonal operator T on V and a unique translation g on V such that $f = T \circ g$.

24. **Theorem 6.23 :** Let T be an orthogonal operator on $\mathbb{R}^2$, and let $A = [T]_\beta$, where β is the standard ordered basis for $\mathbb{R}^2$. Then exactly one of the following conditions is satisfied:

    (a) T is a rotation, and $\det(A) = 1$.
    (b) T is a reflectioni about a line through the origin, and $\det(A) = -1$.

    Let T and U be orthogonal operators on $\mathbb{R}^2$. Use Theorem 6.23 to prove the following results.

    (a) If T and U are both reflections about lines through the origin, then UT is a rotation.
    (b) If T is a rotation and U is a reflection about a line through the origin, then both UT and TU are reflections about lines through the origin.

25. Suppose that T and U are reflections of $\mathbb{R}^2$ about the respective lines L and L' through the origin and that ϕ and ψ are the angles from the positive x-axis to L and L', respectively. By Exercise 24. UT is a rotation. Find its angle of rotation.

26. Suppose that T and U are orthogonal operators on $\mathbb{R}^2$ such that T is the rotation by the angle ϕ and U is the reflection about the line L through the origin. Let ψ be the angle from the positive x-axis to L. By Exercise 24, both UT and TU are reflections about lines L_1 and L_2, respectively, through the origin.

    (a) Find the angle θ from the positive x-axis to L_1.
    (b) Find the angle θ from the positive x-axis to L_2.

27. Find new coordinates x', y' so that the following quadratic forms can be written as $\lambda_1(x')^2 + \lambda_2(y')^2$.

    (a) $x^2 + 4xy + y^2$
    (b) $2x^2 + 2xy + 2y^2$
    (c) $x^2 - 12xy - 4y^2$
    (d) $3x^2 + 2xy + 3y^2$
    (e) $x^2 - 2xy + y^2$

28. Consider the expression X^tAX, where $X^t = (x, y, z)$ and $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find a change of coordinates x', y', z' so that the preceding expression is of the form

$$\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2.$$

29. *QR-Factorization*. Let $w_1, w_2, \ldots, w_n$ be linearly independent vectors in F^n, and let $v_1, v_2, \ldots, v_n$ be the orthogonal vectors obtained from $w_1, w_2, \ldots, w_n$ by the Gram Schmidt process. Let $u_1, u_2, \ldots, u_n$ be the orthonormal basis obtained by normalizing the v_i's.

(a) Solving

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$$

for w_k in terms of u_k, show that

$$w_k = \|v_k\| u_k + \sum_{j=1}^{k-1} \langle w_k, u_j \rangle u_j \quad (1 \le k \le n).$$

(b) Let A and Q denote the $n \times n$ matrices in which the kth columns are w_k and u_k, respectively. Define $R \in M_{n\times n}(F)$ by

$$R_{jk} = \begin{cases} \|v_j\| & \text{if } j = k \\ \langle w_k, u_j \rangle & \text{if } j < k \\ 0 & \text{if } j > k. \end{cases}$$

Prove $A = QR$.

(c) Compute Q and R as in (b) for the 3×3 matrix whose columns are the vectors $(1, 1, 0)$, $(2, 0, 1)$, and $(2, 2, 1)$.

(d) Since Q is unitary [orthogonal] and R is upper triangular in (b), we have shown that every invertible matrix is the product of a unitary [orthogonal] matrix and an upper triangular matrix. Suppose that $A \in M_{n\times n}(F)$ is invertible and $A = Q_1R_1 = Q_2R_2$, where $Q_1, Q_2 \in M_{n\times n}(F)$ are unitary and $R_1, R_2 \in M_{n\times n}(F)$ are upper triangular. Prove that $D = R_2R_1^{-1}$ is a unitary diagonal matrix.

Hint: Use the fact that a matrix that is both unitary and upper triangular must be a diagonal matrix.

(e) The QR factorization described in (b) provides an orthogonalization method for solving a linear system $Ax = b$ when A is invertible. Decompose A to QR, by the Gram Schmidt process or other means, where Q is unitary and R is upper triangular. Then $QRx = b$, and hence $Rx = Q^*b$. This last system can be easily solved since R is upper triangular. [1]

Use the orthogonalization method and (c) to solve the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_1 + 2x_3 &= 11 \\ x_2 + x_3 &= -1. \end{aligned}$$

30. Suppose that β and γ are ordered bases for an n-dimensional real [complex] inner product space V. Prove that if Q is an orthogonal [unitary] $n \times n$ matrix that changes γ-coordinates into β-coordinates, then β is orthonormal if and only if γ is orthonormal.

The following definition is used in Exercises 31 and 32.

Definition. Let V be a finite-dimensional complex [real] inner product space, and let u be a unit vector in V. Define the **Householder** operator $H_u : V \to V$ by $H_u(x) = x - 2\langle x, u \rangle u$ for all $x \in V$.

[1] At one time, because of its great stability, this method for solving large systems of linear equations with a computer was being advocated as a better method than Gaussian elimination even though it requires about three times as much work. (Later, however, J. H. Wilkinson showed that if Gaussian elimination is done "properly,"then it is nearly as stable as the orthogonalization method.)

31. Let H_u be a Householder operator on a finite-dimensional inner product space V. Prove the following results.

    (a) H_u is linear.

    (b) $H_u(x) = x$ if and only if x is orthogonal to u.

    (c) $H_u(u) = -u$.

    (d) $H_u^* = H_u$ and $H_u^2 = 1$, and hence H_u is a unitary [orthogonal] operator on V.

    (Note: If V is a real inner product space, then H_u is a reflection.)

32. Let V be a finite-dimensional inner product space over F. Let x and y be linearly independent vectors in V such that $\|x\| = \|y\|$.

    (a) If $F = \mathbb{C}$, prove that there exists a unit vector u in V and a complex number θ with $|\theta| = 1$ such that $H_u(x) = \theta y$.
    Hint: Choose θ so that $\langle x, \theta y \rangle$ is real, and set $u = \frac{1}{\|x - \theta y\|}(x - \theta y)$.

    (b) If $F = \mathbb{R}$, prove that there exists a unit vector u in V such that $H_u(x) = y$.

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